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REAL HYPERSURFACES CONTAINED IN ABELIAN VARIETIES

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1. In a recent note of these PROCEEDINGS (April, 1919), I showed that an abelian variety of genus p and rank one, V_p , is birationally transformable into a real one if and only if it possesses $2p$ independent linear cycles $\gamma_1, \gamma_2, \dots, \gamma_{2p}$, with respect to which p integrals of the first kind have a period matrix of type $\Omega = \parallel \omega_{h,1}, \dots, \omega_{h,p}; i\omega_{h,p+1}, \dots, i\omega_{h,2p} \parallel$; ($h = 1, 2, \dots, p$), the (ω) 's being real. I propose now to investigate the number ρ' of algebraically distinct real hypersurfaces which V_p , if real, may have. This number $\rho' \leq \rho$, Picard number of V_p , may also be defined as the maximum number of real hypersurfaces which cannot be logarithmic singularities of a simple integral of the third kind.

2. In a general way V_p be an abelian variety of rank one, real or not, with the independent linear cycles $\gamma_1, \gamma_2, \dots, \gamma_{2p}$. By associating γ_μ with γ_ν we obtain a superficial cycle (μ, ν) and any other depends upon those of this type. In particular denoting by (A^{p-1}) the two dimensional cycle formed by A^{p-1} , curve of intersection of $p-1$ algebraic hypersurfaces of the same continuous system as a given one A , we have

$$m(A^{p-1}) \sim \sum_1^{2p} \mu, \nu \ m_{\mu, \nu} (\mu, \nu), (m_{\mu, \nu} \text{ integer} = -m_{\nu, \mu}). \quad (1)$$

It may be shown that if no integral of the first kind is constant on A the alternate form

$$\sum m_{\mu, \nu} x_\mu y_\nu \quad (2)$$

is a principal form of Ω as defined by Scorza (Palermo Rendic., 1916), and conversely to a principal form (2) corresponds an algebraic hypersurface A . Moreover to algebraically distinct hypersurfaces correspond linearly independent principal forms from which follows at once $\rho = 1 + k$, where k is Scorza's index of singularity for Ω .

3. Let us now assume V_p real. A real hypersurface A of V_p is transformed into itself by T , transformation of the variety which permutes its pairs of conjugate points and this property is characteristic for A . It may be shown that there are real curves A^{p-1} ;—let the one of No. 2 be one of them, and α its real line (locus of its real points). A small oriented circuit tangent to α in (A^{p-1}) is transformed by T into one of opposite orientation, for in the neighborhood of α , T behaves like an ordinary plane symmetry. It follows that T transforms the superficial cycle (A^{p-1}) into its opposite. Taking into account the fact that this cycle is a two sided manifold and also

the effect of T upon the linear cycles γ_μ of No. 1, we find at once that all the m 's not of the type $m'_{\mu, p+\nu}$, ($\mu, \nu \leq p$) are equal to zero, hence ρ' is equal to the number of independent forms of type.

$$\sum_1^p \mu, \nu \ m'_{\mu, p+\nu} (x_\mu y_{p+\nu} - x_{p+\nu} y_\mu) \quad (3)$$

which belong to Ω .

If V_p is pure $\rho' \leq p$, for otherwise Ω would possess a degenerate form (3). This is to be contrasted with Scorza's result $1 + k \leq 2p - 1$, or $\rho \leq 2p - 1$ if Ω is pure.

4. Assuming $\rho' = 2$ let L, L' , be the matrices formed by the determinants of two forms (3). They are both of type

$$\left\| \begin{array}{c|c} 0 & \Delta \\ \hline \Delta' & 0 \end{array} \right\|$$

where each square represents a matrix with p rows and columns, the matrices in the main diagonal having only zeroes for terms. As $L^{-1} L'$ is of the form

$$\left\| \begin{array}{c|c} \Delta & 0 \\ \hline 0 & \Delta' \end{array} \right\| = \| a_{\mu\nu} \|, \quad (\mu, \nu = 1, 2, \dots, 2p; a_{\mu, p+\nu} = a_{p+\mu, \nu} = 0),$$

V_p has a complex multiplication defined by

$$\sum_1^p k \lambda_{jk} \omega_{k\mu} = \sum_1^p \nu a_{\mu\nu} \omega_{j\nu}; \quad \sum_1^p j \lambda_{j,k} \omega_{k, p+\mu} = \sum_1^p \nu a_{p+\mu, p+\nu} \omega_{j, p+\nu} \\ (j, \mu = 1, 2, \dots, p),$$

the (λ) 's being necessarily real as they can be replaced by their conjugates. Finally the characteristic equation of this complex multiplication

$$\| a_{\mu\nu} - \epsilon_{\mu\nu} x \| = 0, \quad (\epsilon_{\mu\nu} = 0, \mu \neq \nu; \epsilon_{\mu\mu} = 1)$$

is necessarily reducible and a perfect square if V_p is pure.

5. Let us examine the case of a real hyperelliptic surface of rank one. The number ρ' has then the value 1 or 2, if the surface is pure not elliptic. A fundamental period matrix corresponding to linear cycles forming a minimum base may be reduced to the form

$$\left\| \begin{array}{cc} 1, & 0, \frac{m}{2} + ia, \frac{n}{2} + ib \\ 0, & 1/\delta, \frac{n}{2} + ib, \frac{r}{2} + ic \end{array} \right\|,$$

where m, n, r, δ , are positive integers and $ac - b^2 > 0$. If $\gamma'_1, \gamma'_2, \dots, \gamma'_{2p}$, are the corresponding linear cycles those of No. 1 are given by

$$\gamma_1 = \gamma'_1, \gamma_2 = \gamma'_2, \gamma_3 = 2\gamma'_3 - m\gamma'_1 - n\delta\gamma'_2, \gamma_4 = 2\gamma'_4 - n\gamma'_1 - r\delta\gamma'_2,$$

and in general $\rho = \rho' = 1$, unless there is a singular relation as defined by G. Humbert. If the surface is not elliptic this relation can only be of type

$$\lambda a + \mu b + \gamma \delta c = 0, (\lambda, \mu, \nu, \text{integers}) \quad (3)$$

and there can only be one such relation. In this case $\rho = \rho' = 2$, and the condition of existence becomes now, assuming as we may, $\nu > 0$,

$$\lambda a^2 + \mu ab - \nu \delta b^2 > 0$$

which assures us of the effective existence of the surface. If there are two singular relations such as (5) the surface is elliptic and $\rho = \rho' = 3$.

In addition to (5) there may be in the non-elliptic case as well as in the other a singular relation independent of (5) and reducible to the form

$$\lambda (b^2 - ac) + \mu = 0, (\lambda, \mu, \text{positive integers})$$

and then $\rho - \rho' = 1$, both cases being realizable. Thus there are six distinct types of real hyperelliptic surfaces for which ρ, ρ' have the values: (1, 1), (1, 2), (2, 2), (2, 3), (3, 3), (3, 4), the last three corresponding to elliptic cases.